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JOINT UNIDIRECTIONAL MOTION OF TWO VISCOUS HEAT-CONDUCTING FLUIDS IN A TUBE

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This paper studies an invariant solution of the problem of joint motion of two heat-conducting viscous immiscible fluids which have a common interface in a cylindrical tube under an unsteady pressure gradient. The problem reduces to a coupled initial-boundary-value problem for parabolic equations. A priori estimates of velocity and temperature perturbations are obtained. The steady state of the system is determined, and it is proved that if, in one of the fluids, the pressure gradient rapidly approaches zero, the perturbations of all quantities tend to zero. It is shown that if the pressure gradient has a nonzero limit, the solution reaches a steady state. In this case, the velocity field in the limit is the same as in conjugate Poiseuille flow, and the temperature is represented as a polynomial of the fourth order on the radial coordinate.

Key words: *viscous heat-conducting fluid, interface, steady-state flow.*

1. Basic Equations and Boundary Conditions. As is known, determining the velocity field in unidirectional motion of a viscous fluid in a round tube reduces to solving the linear initial-boundary-value problem for a parabolic equation [1, 2]. In this case, the solution is found in the form of finite formulas (for example, steady-state Poiseuille flow) or in the form of series in Bessel functions. In the present paper, we study the coupled initial-boundary-value problem for unidirectional motion of two viscous heat-conducting fluids with a common interface in a cylindrical tube under an unsteady pressure gradient. We note that, for the case of plane layers, this problem was studied in [3].

In the absence of body forces, the system of equations describing axisymmetric motion of viscous heat-conducting fluids is written in cylindrical coordinates in the form

$$u_t + uu_r + wu_z - v^2/r = -p_r/\rho + \nu(\Delta u - u/r^2),$$
$$w_t + uw_r + ww_z = -p_z/\rho + \nu\,\Delta w, \tag{1.1}$$
$$u_r + u/r + w_z = 0, \qquad \theta_t + u\theta_r + w\theta_z = \chi\,\Delta\theta.$$

Here $u(r,z,t)$ and $w(r,z,t)$ are projections of the velocity onto the r and z axes, respectively, $p(r,z,t)$ is the pressure, $\theta(r,z,t)$ is the deviation of the temperature from its equilibrium value θ_0, ν, ρ, and χ are the kinematic viscosity, density, and thermal diffusivity, respectively, and $\Delta = \partial^2/\partial r^2 + r^{-1}\,\partial/\partial r + \partial^2/\partial z^2$ is the Laplacian.

System (1.1) admits the one-parameter subgroup of continuous transformations corresponding to the operator $\partial/\partial z + A\,\partial/\partial\theta - \rho f(t)\,\partial/\partial p$ (A is a constant; $f(t) \in C^\infty$ is an arbitrary function). The invariant solution should be sought in the form

$$u = u(r,t), \qquad w = w(r,t), \qquad p = -\rho f(t)z + D(r,t), \qquad \theta = Az + T(r,t).$$

The fourth equation of system (1.1) implies that $u(r,t) = g(t)/r$ with an arbitrary function $g(t)$ which is then set equal to zero. Then, from the first equation in (1.1), it follows that D is a function of only time. In view of the above assumptions, the solution is represented as

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$$u = 0, \qquad w = w(r,t), \qquad p = -\rho f(t) z + D(t), \qquad \theta = Az + T(r,t). \tag{1.2}$$

We use solution (1.2) to describe unidirectional motion of two viscous heat-conducting fluids in a round cylindrical tube of radius b. Suppose one of the fluids occupies a region $0 \le r \le a$, $|z| < \infty$ and the other occupies a cylindrical layer $a \le r \le b$, $|z| < \infty$; $w_j(r,t)$ is the axial velocity ($j = 1, 2$), $p_j = -\rho_j f_j(t) z + D_j(t)$ is the pressure, and $\theta_j = A_j z + T_j(r,t)$ is the temperature distribution. Let us assume that the surface tension coefficient on the interface $r = a$ depends linearly on temperature:

$$\sigma(\theta) = \sigma_0 - æ_1(\theta - \theta_0) \tag{1.3}$$

($æ_1 > 0$ is a constant).

On the interface, the following conditions are satisfied [4]:

$$(P_2 - P_1)\boldsymbol{n} = 2\sigma H \boldsymbol{n} + \nabla_\Gamma \sigma; \tag{1.4}$$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2, \qquad \theta_1 = \theta_2, \qquad k_1 \frac{\partial \theta_1}{\partial n} = k_2 \frac{\partial \theta_2}{\partial n}. \tag{1.5}$$

Here

$$P_j = \begin{pmatrix} -p_j & 0 & \mu_j w_{jr} \\ 0 & -p_j & 0 \\ \mu_j w_{jr} & 0 & -p_j \end{pmatrix}$$

is the tension coefficient in the representation of solution (1.2), H is the average curvature of the interface, $\nabla_\Gamma = \boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n} \cdot \nabla)$ is the surface gradient, and k_j are the thermal conductivities of the fluids. The equalities $\boldsymbol{n} = (1, 0, 0)$ and $H = \sigma/(2a)$ imply that, for $r = a$, relation (1.4) is equivalent to the two relations

$$p_2 - p_1 = \frac{\sigma}{a}, \qquad \mu_2 w_{2r} - \mu_1 w_{1r} = \frac{\partial \sigma}{\partial z}. \tag{1.6}$$

The first of these relations is the Laplace formula: the pressure difference on the curvilinear surface is counterbalanced by capillary forces. The second relation implies that the difference of shear stresses is counterbalanced by the thermocapillary forces due to variation in the surface tension along the interface.

It is easy to show that, for $r = a$, equalities (1.5) lead to the boundary conditions

$$w_1(a,t) = w_2(a,t), \qquad T_1(a,t) = T_2(a,t), \qquad k_1 \frac{\partial T_1(a,t)}{\partial r} = k_2 \frac{\partial T_2(a,t)}{\partial r} \tag{1.7}$$

and, in addition,

$$A_1 = A_2 \equiv A. \tag{1.8}$$

In turn, in view of (1.2), (1.3), and (1.8) from (1.6) we obtain

$$f_2(t) = \rho f_1(t) - h/(\rho_2 a), \qquad h \equiv -æ_1 A, \qquad \rho = \rho_1/\rho_2,$$
$$D_2(t) = D_1(t) + [\sigma_0 - æ_1(T_1(a,t) - \theta_0)]/a; \tag{1.9}$$

$$\mu_2 w_{2r}(a,t) - \mu_1 w_{1r}(a,t) = h. \tag{1.10}$$

On the solid wall ($r = b$), we impose the boundary conditions

$$w_2(b,t) = 0, \qquad T_2(b,t) = 0. \tag{1.11}$$

On the symmetry axis, boundedness conditions are imposed:

$$|w_1(0,t)| < \infty, \qquad |T_1(0,t)| < \infty. \tag{1.12}$$

In view of representation (1.2) of the solutions of system (1.1), the sought functions $w_j(r,t)$ and $T_j(r,t)$ are expressed as:

$$w_{jt} = f_j(t) + \nu_j(w_{jrr} + w_{jr}/r); \tag{1.13}$$

$$T_{jt} = \chi_j(T_{jrr} + T_{jr}/r) - Aw_j. \tag{1.14}$$

In Eqs. (1.13) and (1.14) for $j = 1$, the variable r varies in the range of $0 < r < a$, and for $j = 2$, it varies in the range $a < r < b$. Thus, it is required to solve problem (1.13), (1.14) with boundary conditions (1.7) and (1.10)–(1.12) and the initial conditions

$$w_j(r,0) = 0, \qquad T_j(r,0) = 0. \tag{1.15}$$

The pressure is found form the formulas $p_j(r,t) = -\rho_j f_j(t)z + D_j(t)$, and the functions $f_1(t)$, $f_2(t)$, and $D_1(t)$, and $D_2(t)$ are linked by equalities (1.9) and, in addition, the functions $f_1(t)$ and $D_1(t)$ should be specified.

Remark 1. At $t = 0$, due to the first initial condition (1.15) boundary condition (1.10) undergoes a discontinuity of the first kind.

Next, it is assumed that $h \equiv 0$, i.e., the surface tension does not depend on temperature and motion occurs only due to the pressure gradient.

2. Steady-State Solution. In the case of steady-state solution, all sought functions do not depend on time. We denote these functions as $w_j^0(r)$ and $T_j^0(r)$. In addition, $f_1(t) = f_1^0 = \text{const}$ and $D_1(t) = D_1^0 = \text{const}$. For $f_2^0 = \rho f_1^0$, the steady-state solution of problem (1.13), (1.14), (1.7), (1.10)–(1.12) has the form

$$w_1^0 = \frac{f_1^0}{4\nu_1}\Big[(b^2-a^2)\mu + a^2\Big(1-\frac{r^2}{a^2}\Big)\Big], \qquad w_2^0 = \frac{f_1^0 b^2\mu}{4\nu_1}\Big(1-\frac{r^2}{b^2}\Big), \qquad \mu = \frac{\mu_1}{\mu_2},$$

$$T_1^0 = \frac{Ar^2 f_1^0}{16\chi_1\nu_1}\Big(a^2 + \mu(b^2-a^2) - \frac{r^2}{4}\Big) + C_1, \qquad T_2^0 = \frac{Ar^2\rho f_1^0}{16\chi_2\nu_2}\Big(b^2 - \frac{r^2}{4}\Big) + C_2\ln r + C_3, \tag{2.1}$$

where

$$C_1 = \frac{a^4 f_1^0 A}{8\nu_1\chi_1}\Big\{-\frac{3\mu\chi}{8}\Big(\frac{b}{a}\Big)^4 + \Big[k\Big(\frac{1}{2}+\mu\,\frac{b^2-1}{a^2}\Big) + \mu\chi\Big(\frac{1}{2}-\frac{b^2}{a^2}\Big)\Big]\ln\frac{a}{b} - \frac{\mu}{2}\Big[\frac{b^2}{a^2} - 1 + \chi\Big(\frac{1}{4}-\frac{b^2}{a^2}\Big)\Big] - \frac{3}{8}\Big\},$$

$$C_2 = \frac{a^2 f_1^0 A}{8\chi_1}\Big[\frac{a^2}{\nu_1} + \frac{(b^2-a^2)\rho}{\nu_2} + \frac{\chi\rho}{\nu_2}\Big(\frac{a^2}{2} - b^2\Big)\Big],$$

$$C_3 = \frac{f_1^0 A}{8\chi_1}\Big\{-\frac{3\chi b^4\rho}{8\nu_2} - a^2\Big[k\Big(\frac{a^2}{2\nu_1} + \frac{b^2-a^2}{\nu_2}\Big) + \frac{\chi}{\nu_2}\Big(\frac{a^2}{2} - b^2\Big)\rho\Big]\ln b\Big\},$$

$$\chi = \frac{\chi_1}{\chi_2}, \qquad k = \frac{k_1}{k_2}, \qquad \mu = \frac{\mu_1}{\mu_2}, \qquad \rho = \frac{\rho_1}{\rho_2}.$$

3. *A Priori* Estimate of the Velocity Field. It should be noted that for the specified function $f_1(t)$, the problems for w_j, T_j $(j = 1, 2)$ are solved sequentially [see (1.7)–(1.15)]. The initial-boundary-value problem for the sought functions $w_j(r,t)$ has the form

$$w_{1t} = \nu_1(w_{1rr} + w_{1r}/r) + f_1(t), \qquad 0 < r < a; \tag{3.1}$$

$$w_{2t} = \nu_2(w_{2rr} + w_{2r}/r) + \rho f_1(t), \qquad a < r < b; \tag{3.2}$$

$$w_1(r,0) = 0, \qquad w_2(r,0) = 0; \tag{3.3}$$

$$|w_1(0,t)| < \infty; \tag{3.4}$$

$$w_2(b,t) = 0; \tag{3.5}$$

$$w_1(a,t) = w_2(a,t), \qquad \mu_2 w_{2r}(a,t) - \mu_1 w_{1r}(a,t) = 0. \tag{3.6}$$

Multiplying Eq. (3.1) by $\rho_1 r w_1$ [or Eq. (3.2) by $\rho_2 r w_2$] and integrating the result in parts with respect to r from zero to a (or from a to b) using the boundedness (3.4) and no-slip (3.5) conditions, we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\rho_1\int_0^a r w_1^2\,dr = \mu_1 a w_1(a,t) w_{1r}(a,t) - \mu_1\int_0^a r w_{1r}^2\,dr + \rho_1 f_1(t)\int_0^a r w_1\,dr; \tag{3.7}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\rho_2\int_a^b r w_2^2\,dr = -\mu_2 a w_2(a,t) w_{2r}(a,t) - \mu_2\int_a^b r w_{2r}^2\,dr + \rho f_1(t)\int_a^b r w_2\,dr. \tag{3.8}$$

Combining Eqs. (3.7) and (3.8), we obtain the relation

$$\frac{\partial}{\partial t}E + \mu_1\int_0^a r w_{1r}^2\,dr + \mu_2\int_a^b r w_{2r}^2\,dr = \rho_1 f_1(t)\Big(\int_0^a r w_1\,dr + \int_a^b r w_2\,dr\Big), \tag{3.9}$$

where $E(t)$ is the kinetic energy of two layers:

$$E(t) = \frac{1}{2}\Big(\rho_1\int_0^a r w_1^2\,dr + \rho_2\int_a^b r w_2^2\,dr\Big). \tag{3.10}$$

Equation (3.9) was derived with boundary conditions (3.6) taken into account.

Equation (3.9) implies the uniqueness of the solution of problem (3.1)–(3.6) because, for $f_1 = 0$ and $f_2 = 0$ we obtain $E(t) = 0$ follows and, hence, $w_1 = w_2 = 0$.

The expression in brackets on the right of (3.9), admits the upper bound estimate

$$\Big(\int_0^a r\,dr\Big)^{1/2}\Big(\int_0^a r w_1^2\,dr\Big)^{1/2} + \Big(\int_a^b r\,dr\Big)^{1/2}\Big(\int_a^b r w_2^2\,dr\Big)^{1/2} \le C_1\sqrt{E(t)} \tag{3.11}$$

with the constant

$$C_1 = \sqrt{2}\,\max\Big(\frac{a}{\sqrt{\rho_1}}, \frac{\sqrt{b^2-a^2}}{\sqrt{\rho_2}}\Big), \tag{3.12}$$

because for nonnegative x and y, we have $\sqrt{x}+\sqrt{y} \le \sqrt{2(x+y)}$.

The solution of problem (3.1)–(3.6) also satisfies the identity

$$\rho_1\int_0^t\int_0^a r\Big[w_{1t}^2 + \nu_1^2\Big(w_{1rr} + \frac{1}{r}\,w_{1r}\Big)^2\Big]\,dr\,dt + \rho_2\int_0^t\int_a^b r\Big[w_{2t}^2 + \nu_2^2\Big(w_{2rr} + \frac{1}{r}\,w_{2r}\Big)^2\Big]\,dr\,dt$$

$$+\,\mu_1\int_0^a r w_{1r}^2\,dr + \mu_2\int_a^b r w_{2r}^2\,dr = \frac{\rho_1}{2}\,[a^2 + \rho(b^2-a^2)]\int_0^t f_1^2(t)\,dt, \tag{3.13}$$

which is derived by squaring the equations $w_{jt} - \nu_j(w_{jrr} + w_{jr}/r) = f_j(t)$ and their integration with respect to r and time. Therefore, if the integral

$$\int_0^\infty f_1^2(t)\,dt = C_2^2 \tag{3.14}$$

converges, we have

$$\int_0^a r w_{1r}^2\,dr \le \frac{[a^2+\rho(b^2-a^2)]C_2^2}{2\nu_1}, \qquad \int_a^b r w_{2r}^2\,dr \le \frac{\rho_1[a^2+\rho(b^2-a^2)]C_2^2}{2\mu_2}. \tag{3.15}$$

Next, we require the following lemma.

Lemma 1. *The inequality*

$$\int_0^a rw_1^2\,dr + \int_a^b rw_2^2\,dr \le M_0\Big(\mu_1\int_0^a rw_{1r}^2\,dr + \mu_2\int_a^b rw_{2r}^2\,dr\Big) \tag{3.16}$$

holds, where the constant M_0 *does not depend on* w_j *and is a solution of the variational problem*

$$M_0 = \sup_{v_1,v_2\in V}\Big[\Big(\int_0^a rv_1^2\,dr + \int_a^b rv_2^2\,dr\Big)\Big/\Big(\mu_1\int_0^a rv_{1r}^2\,dr + \mu_2\int_a^b rv_{2r}^2\,dr\Big)\Big].$$

The set V *is a subspace of* $W_2^1(r;0,a)\times W_2^1(r;a,b)$*, and* v_1 *and* v_2 *satisfy the boundary conditions* (3.3)–(3.6).

A proof is given in [5]. We denote $a_1 = b/a$, $a_2 = \sqrt{\mu_1/\mu_2}$, and $x = a/\sqrt{\mu_1 M_0}$ Then, M_0 (or, more precisely, x) is a root of the transcendental equation

$$J_0(x)[J_1(a_2x)Y_0(a_1a_2x) - J_0(a_1a_2x)Y_1(a_2x)]$$

$$+ a_2J_1(x)[J_0(a_1a_2x)Y_0(a_2x) - J_0(a_2x)Y_0(a_1a_2x)] = 0, \tag{3.17}$$

where J_k and Y_k $(k = 0,1)$ are Bessel functions of the first and second kind. Equation (3.17) has positive roots [5]. Let $x_0 = x_0(a_1,a_2)$ be the smallest positive root of Eq. (3.17). Then, $M_0 = a^2/(\mu_1x_0^2)$ is the exact value of the constant in inequality (3.16).

Using inequality (3.16), we obtain the inequality

$$\mu_1\int_0^a rw_{1r}^2\,dr + \mu_2\int_a^b rw_{2r}^2\,dr \ge 2\delta E, \tag{3.18}$$

where

$$\delta = \frac{1}{M_0}\,\min\Big(\frac{1}{\rho_1},\frac{1}{\rho_2}\Big). \tag{3.19}$$

Equations (3.9)–(3.12), (3.18), and (3.19) lead to the inequality $dE/dt + 2\,\delta E \le C_1\rho_1|f_1(t)|\sqrt{E}$, from which we obtain

$$E \le \frac{C_1^2\rho_1^2}{4}\Big(\int_0^t |f_1(\tau)|\,e^{\delta\tau}\,d\tau\Big)^2 e^{-2\delta t}.$$

Hence, if the integral

$$\int_0^\infty |f_1(\tau)|\,e^{\delta\tau}\,d\tau = C_3 \tag{3.20}$$

converges, then,

$$\int_0^a rw_1^2\,dr \le \frac{\rho_1C_1^2C_3^2\,e^{-2\delta t}}{2}, \qquad \int_a^b rw_2^2\,dr \le \frac{\rho_1^2C_1^2C_3^2\,e^{-2\delta t}}{2\rho_2}. \tag{3.21}$$

Remark 2. The convergence of integral (3.20) implies the convergence of integral (3.14).

We investigate the behavior of the function $w_j(r,t)$ as $t\to\infty$. In view of estimates (3.15) and (3.21) for $w_2(r,t)$, the following inequality is valid:

$$w_2^2(r,t) = \Big|\int_r^b (w_2^2)_r\,dr\Big| \le 2\int_a^b |w_2|\,|w_{2r}|\,dr = 2\int_a^b \frac{1}{r}\sqrt{r}\,|w_2|\sqrt{r}\,|w_{2r}|\,dr$$

$$\le \frac{2}{a}\Big(\int_a^b rw_2^2\,dr\Big)^{1/2}\Big(\int_a^b rw_{2r}^2\,dr\Big)^{1/2} \le C_4\,e^{-\delta t},$$

where

$$C_4 = \rho_1 C_1 C_2 C_3 \sqrt{(\rho/\mu_2)[a^2 + \rho(b^2 - a^2)]/a}\,.$$

Hence,

$$|w_2(r,t)| \le \sqrt{C_4}\,\mathrm{e}^{-\delta t/2} \tag{3.22}$$

for any $r \in [a,b]$. However, by similar reasoning, it is impossible to obtain an estimate for $|w_1(r,t)|$. In this case, from (3.22) and the first equality in (3.6), we have

$$|w_1(a,t)| = |w_2(a,t)| \le \sqrt{C_4}\,\mathrm{e}^{-\delta t/2}\,.$$

If the function $w_1(a,t)$ is considered known, the solution of the initial-boundary-value problem

$$w_{1t} = \nu_1(w_{1rr} + w_{1r}/r) + f_1(t), \qquad 0 < r < a,$$

$$w_1(r,0) = 0, \qquad |w_1(0,t)| < \infty, \qquad w_1(r,t)\Big|_{r=a} = w_1(a,t)$$

for the function $w_1(r,t)$ has the following form [6, p. 74]:

$$w_1(r,t) = \frac{2\nu_1}{a^2}\sum_{n=1}^{\infty}\frac{\mu_n J_0(\mu_n r/a)}{J_1(\mu_n)}\int_0^t w_1(a,\tau)\,\mathrm{e}^{-\nu_1\mu_n^2(t-\tau)/a^2}\,d\tau$$

$$+\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\mu_n r/a)}{\mu_n J_1(\mu_n)}\int_0^t f_1(\tau)\,\mathrm{e}^{-\nu_1\mu_n^2(t-\tau)/a^2}\,d\tau, \tag{3.23}$$

Here μ_n are zeroes of the Bessel functions J_0.

The integrals in (3.23) satisfy the estimates

$$\Big|\int_0^t w_1(a,\tau)\,\mathrm{e}^{-\nu_1\mu_n^2(t-\tau)/a^2}\,d\tau\Big| \le \frac{2a^2\sqrt{C_4}}{2\nu_1\mu_n^2 - a^2\delta}\Big(\mathrm{e}^{-\delta t/2} - \mathrm{e}^{-\nu_1\mu_n^2 t/a^2}\Big),$$

$$\Big|\int_0^t f_1(\tau)\,\mathrm{e}^{-\nu_1\mu_n^2(t-\tau)/a^2}\,d\tau\Big| \le \frac{a^2 N}{\nu_1\mu_n^2 - a^2\delta}\Big(\mathrm{e}^{-\delta t} - \mathrm{e}^{-\nu_1\mu_n^2 t/a^2}\Big), \tag{3.24}$$

because, in view of (3.20), we have $|f_1(t)|\,\mathrm{e}^{\delta t} \leqslant N$ (N is a constant).

From (3.23) and (3.24), it follows that

$$|w_1(r,t)| \le N_1 \begin{cases} \mathrm{e}^{-\delta t/2}, & \delta \le 2\mu_1^2\nu_1/a^2, \\ \mathrm{e}^{-\mu_1^2\nu_1 t/a^2}, & \delta > 2\mu_1^2\nu_1/a^2. \end{cases} \tag{3.25}$$

Here N_1 is a constant and μ_1 the first root of the equation $J_0(\mu) = 0$ (nondynamic viscosity). We note that the series in (3.23) converge absolutely in view of inequalities (3.24). Thus, we proved the following theorem.

Theorem 1. *If condition* (3.20) *is satisfied, the solution of the initial-boundary-value problem* (3.1)–(3.6) *tends to the zero solution, and estimates* (3.22) *and* (3.25) *are valid, which are uniform in the intervals* $[a,b]$ *and* $[0,a]$.

In other words, if the pressure gradient in one of the fluids rapidly approaches zero, there is deceleration of the fluids due to viscous friction [see (3.22) and (3.25)].

4. Solution Using the Laplace Transform. We set

$$\tilde{w}_j(r,p) = \int_0^{\infty} w_j(r,t)\,\mathrm{e}^{-pt}\,dt. \tag{4.1}$$

Then, problem (3.1)–(3.6) reduces to the boundary-value problem for the ordinary differential equations

$$\tilde{w}_{1rr} + \tilde{w}_{1r}/r - p\tilde{w}_1/\nu_1 = -\tilde{f}_1(p)/\nu_1, \qquad 0 < r < a,$$
$$\tilde{w}_{2rr} + \tilde{w}_{2r}/r - p\tilde{w}_2/\nu_2 = -\rho\tilde{f}_1(p)/\nu_2, \qquad a < r < b; \tag{4.2}$$

$$\tilde{w}_1(a,p) = \tilde{w}_2(a,p), \qquad \tilde{w}_2(b,p) = 0; \tag{4.3}$$

$$\mu_2\tilde{w}_{2r}(a,p) = \mu_1\tilde{w}_{1r}(a,p). \tag{4.4}$$

In view of the inequalities $|\tilde{w}_1(0,p)| < \infty$, the general solutions of Eqs. (4.2) have the form

$$\tilde{w}_1 = C_1 I_0(\sqrt{p/\nu_1}\, r) + \tilde{f}_1(p)/p; \tag{4.5}$$

$$\tilde{w}_2 = C_2 I_0(\sqrt{p/\nu_2}\, r) + C_3 K_0(\sqrt{p/\nu_2}\, r) + \tilde{f}_2(p)/p. \tag{4.6}$$

From boundary conditions (4.3) and (4.4), we have

$$C_1 = \frac{\tilde{f}_1}{p\Delta}\begin{vmatrix} -\rho & I_0(z) & K_0(z) \\ \rho - 1 & -I_0(y) & -K_0(y) \\ 0 & -I_1(y) & K_1(y) \end{vmatrix}; \tag{4.7}$$

$$C_2 = \frac{\tilde{f}_1}{p\Delta}\begin{vmatrix} 0 & -\rho & K_0(z) \\ I_0(x) & \rho - 1 & -K_0(y) \\ \mu I_1(x)/\sqrt{\nu} & 0 & K_1(y) \end{vmatrix}; \tag{4.8}$$

$$C_3 = \frac{\tilde{f}_1}{p\Delta}\begin{vmatrix} 0 & I_0(z) & -\rho \\ I_0(x) & -I_0(y) & \rho - 1 \\ \mu I_1(x)/\sqrt{\nu} & -I_1(y) & 0 \end{vmatrix}; \tag{4.9}$$

$$\Delta = \begin{vmatrix} 0 & I_0(z) & K_0(z) \\ I_0(x) & -I_0(y) & -K_0(y) \\ \mu I_1(x)/\sqrt{\nu} & -I_1(y) & K_1(y) \end{vmatrix}, \tag{4.10}$$

where $\mu = \mu_1/\mu_2$, $\nu = \nu_1/\nu_2$, $x = a\sqrt{p/\nu_1}$, $y = a\sqrt{p/\nu_2}$, $z = b\sqrt{p/\nu_2}$, and I_j and K_j are Bessel functions of the first and third kind of an imaginary argument.

Using the explicit formulas (4.5)–(4.10), it is possible to show that $\lim\limits_{t\to\infty} w_j(r,t) = \lim\limits_{p\to 0} p\tilde{w}_j(r,p) = w_j^0(r)$, where $w_j^0(r)$ is the steady-state velocity field defined in (2.1). Indeed, since for small t, the representations $I_0(t) \approx 1 + t^2/4$, $K_0(t) \approx -\ln t/2$, and $I_1(t) \approx t/2 + t^3/16$, $K_1(t) \approx 1/t + (t\ln t)/2$ are valid, we have

$$\Delta(p) \sim -\Big(1 + \frac{x^2 + z^2}{4} - \frac{y^2}{2}\,(\rho - 1)\ln\frac{a}{b}\Big)\frac{1}{y} \tag{4.11}$$

as $p \to 0$. Then, from (4.7)–(4.11) after a series of transformations, we obtain

$$C_1 \sim -\tilde{f}_1[1 + (\rho y^2 - \rho z^2 - x^2)/4]/p, \quad C_2 \sim -\rho\tilde{f}_1(1 - z^2/4)/p,$$
$$C_3 \sim -\rho\tilde{f}_1 y^2[x^2 + (\rho - 1)z^2 - \rho y^2]/(8p). \tag{4.12}$$

As $p \to 0$, Eqs. (4.5) and (4.12) lead to the expression

$$p\tilde{w}_1(r,p) \sim -\tilde{f}_1[1 + (\rho(y^2 - z^2) - x^2)/4][1 + pr^2/(4\nu_1)] + \tilde{f}_1$$
$$= p\tilde{f}_1(p)[a^2 - r^2 + \mu(b^2 - a^2)]/(4\nu_1) \to f_1^0[a^2 - r^2 + \mu(b^2 - a^2)]/(4\nu_1),$$

which coincides with the first expression for $w_1^0(r)$ in (2.1). Similarly, we can show that $\lim\limits_{t\to\infty} w_2(r,t) = \lim\limits_{p\to 0} p\tilde{w}_2(r,p) = w_2^0(r)$.

5. Determining the Flow Rate or Pressure Gradient. The fluid flux flowing through the cross section in unit time is equal to

$$Q_1(t) = 2\pi \int_0^a r w_1(r,t)\,dr, \qquad Q_2(t) = 2\pi \int_a^b r w_2(r,t)\,dr. \tag{5.1}$$

In the case of the steady-state solution (2.1), we have

$$Q_1^0 = \pi a^4 f_1^0 [1/2 + (b^2/a^2 - 1)\mu]/(4\nu_1), \qquad \mu = \mu_1/\mu_2; \tag{5.2}$$

$$Q_2^0 = \pi \rho_1 f_1^0 (b^2 - a^2)^2/(8\mu_2). \tag{5.3}$$

The ratio of the flow rates Q_1^0 and Q_2^0 is equal to

$$\frac{Q_1^0}{Q_2^0} = \frac{4[1/2 + \mu(b^2/a^2 - 1)]}{\mu(b^2/a^2 - 1)^2}.$$

Thus, for $b/a = 1.1$ and $\mu = 0.1$, the flow rate Q_1^0 is approximately 500 times larger than the flow rate Q_2^0.

If the flow rate of the first fluid $Q_1(t)$ is specified, it is possible to find the function $\tilde{Q}_1(p)$. At the same time, from the first formula in (5.2) and from (4.5), (4.7), and (4.10), we obtain

$$\tilde{Q}_1(p) = 2\pi \int_0^a r\tilde{w}_1(r,p)\,dr = 2\pi \Big[\frac{\tilde{f}_1(p) a^2}{2p} + C_1 \sqrt{\frac{\nu_1}{p}}\, I_1\Big(\sqrt{\frac{p}{\nu_1}}\, a\Big)\Big]. \tag{5.4}$$

Using the known function $\tilde{Q}_1(p)$, from (5.4) we obtain $\tilde{f}_1(p)$ and from the formula

$$f_1(t) = \frac{1}{2\pi i} \int_{l-i\infty}^{l+i\infty} \mathrm{e}^{pt}\, \tilde{f}_1(p)\,dp$$

we restore the original. Since $f_2(t) = \rho f_1(t)$, for the specified $Q_1(t)$, the velocity fields and pressure gradients in the layers are completely determined. In other words, in this case, it is also possible to solve the inverse problem: using the specified function $Q_1(t)$, to find three functions $w_1(r,t)$, $w_2(r,t)$, and $f_1(t)$ which satisfy the initial-boundary-value problem (3.1)–(3.6) and the first integral condition in (5.1).

6. Determining Temperature Perturbations in the Layers. The initial-boundary-value problem for the functions $T_j(r,t)$ has the form

$$\begin{gathered} T_{1t} = \chi_1(T_{1rr} + T_{1r}/r) - Aw_1(r,t), \qquad 0 < r < a, \\ T_{2t} = \chi_2(T_{2rr} + T_{2r}/r) - Aw_2(r,t), \qquad a < r < b, \\ T_1(a,t) = T_2(a,t), \qquad k_1 T_{1r}(a,t) = k_2 T_{2r}(a,t), \\ T_2(b,t) = 0, \qquad |T_1(0,t)| < \infty, \\ T_1(r,0) = 0, \qquad T_2(r,0) = 0. \end{gathered} \tag{6.1}$$

We note that problem (6.1) coincides with problem (3.1)–(3.6) if we make the change $w_j \leftrightarrow T_j$, $f_j \leftrightarrow -Aw_j$, $\nu_j \leftrightarrow \chi_j$, and $\mu_j \leftrightarrow k_j$. In addition, it is necessary to take into account that $k_j = \chi_j \rho_j c_j$ (c_j are the specific heat capacities). Therefore, introducing the notation

$$E_1(t) = \frac{1}{2}\rho_1 c_1 \int_0^a r T_1^2(r,t)\,dr + \frac{1}{2}\rho_2 c_2 \int_a^b r T_2^2(r,t)\,dr, \tag{6.2}$$

we obtain an equality similar to (3.9)

$$\frac{\partial E_1}{\partial t} + k_1 \int_0^a r T_{1r}^2\,dr + k_2 \int_a^b r T_{2r}^2\,dr = -A\Big(\rho_1 c_1 \int_0^a r w_1 T_1\,dr + \rho_2 c_2 \int_a^b r w_2 T_2\,dr\Big). \tag{6.3}$$

According to Lemma 1, we have

$$k_1 \int_0^a r T_{1r}^2 \, dr + k_2 \int_a^b r T_{2r}^2 \, dr \geq 2\delta_1 E_1, \tag{6.4}$$

where

$$\delta_1 = \frac{1}{M_0} \min\Big(\frac{1}{\rho_1 c_1}, \frac{1}{\rho_2 c_2}\Big) = \frac{1}{M_0} \min\Big(\frac{\chi_1}{k_1}, \frac{\chi_2}{k_2}\Big). \tag{6.5}$$

The expression in brackets on the right of (6.3) admits the upper-bound estimate

$$\Big|\rho_1 c_1 \int_0^a r w_1 T_1 \, dr + \rho_2 c_2 \int_a^b r w_2 T_2 \, dr\Big| \leq \rho_1 c_1 \Big(\int_0^a r w_1^2 \, dr\Big)^{1/2} \Big(\int_0^a r T_1^2 \, dr\Big)^{1/2} + \rho_2 c_2 \Big(\int_a^b r w_2^2 \, dr\Big)^{1/2} \Big(\int_a^b r T_2^2 \, dr\Big)^{1/2}$$

$$\leq 2 \max\Big[\Big(\int_0^a r w_1^2 \, dr\Big)^{1/2}, \Big(\int_a^b r w_2^2 \, dr\Big)^{1/2}\Big] \max\Big(\sqrt{\rho_1 c_1}\,, \sqrt{\rho_2 c_2}\Big) \sqrt{E_1(t)}\,. \tag{6.6}$$

In view of (3.21), from (6.3)–(6.6) we obtain

$$\frac{\partial E_1}{\partial t} + 2\delta_1 E_1 \leq C_5 \sqrt{E_1}\, \mathrm{e}^{-\delta t}, \tag{6.7}$$

where

$$C_5 = \sqrt{2\rho_1}\, C_1 C_2 |A| \max(1, \rho) \max\Big(\sqrt{k_1/\chi_1}\,, \sqrt{k_2/\chi_2}\Big).$$

Since $E_1(0) = 0$, integrating inequality (6.7), we find the estimate $E_1(t)$:

$$E_1(t) \leq \frac{C_5^2}{4} \begin{cases} t^2 \, \mathrm{e}^{-2\delta t}, & \delta_1 = \delta, \\ \Big(\mathrm{e}^{-\delta t} - \mathrm{e}^{-\delta_1 t}\Big)^2 / (\delta_1 - \delta)^2, & \delta_1 \neq \delta. \end{cases} \tag{6.8}$$

Then, Eqs. (6.2) and (6.8) lead to

$$\int_0^a r T_1^2 \, dr \leq \frac{2\chi_1}{k_1} E_1(t), \qquad \int_a^b r T_2^2 \, dr \leq \frac{2\chi_2}{k_2} E_1(t), \tag{6.9}$$

therefore, the squares of the L_2-norms with the weight r of the functions $T_1(r,t)$ and, $T_2(r,t)$ tend exponentially to zero as $t \to \infty$ if equality (3.20) is satisfied.

In the case of problem (6.1), an identity similar to (3.13) holds:

$$\rho_1 c_1 \int_0^t \int_0^a r\Big[T_{1t}^2 + \chi_1^2 \Big(T_{1rr} + \frac{1}{r} T_{1r}\Big)^2\Big] \, dr \, dt + \rho_2 c_2 \int_0^t \int_a^b r\Big[T_{2t}^2 + \chi_2^2 \Big(T_{2rr} + \frac{1}{r} T_{2r}\Big)^2\Big] \, dr \, dt$$

$$+ k_1 \int_0^a r T_{1r}^2 \, dr + k_2 \int_a^b r T_{2r}^2 \, dr = A^2 \Big(\rho_1 c_1 \int_0^t \int_0^a r w_1^2 \, dr \, dt + \rho_2 c_2 \int_0^t \int_a^b r w_2^2 \, dr \, dt\Big). \tag{6.10}$$

From this and from estimates (3.21), we obtain the inequalities

$$\int_0^a r T_{1r}^2 \, dr \leq \frac{C_6}{k_1}, \qquad \int_a^b r T_{2r}^2 \, dr \leq \frac{C_6}{k_2}, \tag{6.11}$$

where

$$C_6 = A^2 C_1^2 C_3^2 \rho_1 (k_1/\chi_1 + \rho k_2/\chi_2)/(4\delta).$$

From (6.9)–(6.11), by analogy with estimate (3.22), we obtain

$$|T_2(r,t)| \le \sqrt{C_7}\,(E_1(t))^{1/4}, \qquad C_7 = (2/a)\sqrt{2\chi_2 C_6/k_2^2}\,. \tag{6.12}$$

To estimate $|T_1(r,t)|$, $r \in [0,a]$, we proceed in the same way as in Sec. 3. The function $T_1(r,t)$ is a solution of the initial-boundary-value problem

$$\begin{gathered} T_{1t} = \chi_1(T_{1rr} + T_{1r}/r) - Aw_1(r,t), \qquad 0 < r < a, \\ T_1(r,0) = 0, \qquad |T_1(0,t)| < \infty, \qquad T_1(a,t) = T_2(a,t), \end{gathered} \tag{6.13}$$

and, according to estimate (6.12), we have

$$|T_1(a,t)| \le \sqrt{C_7}\,(E_1(t))^{1/4}. \tag{6.14}$$

The solution of problem (6.13) [6, p. 74] has the form

$$T_1(r,t) = \frac{2\chi_1}{a^2}\sum_{n=1}^{\infty}\frac{\mu_n J_0(\mu_n r/a)}{J_1(\mu_n)}\int_0^t T_1(a,\tau)\,\mathrm{e}^{-\chi_1\mu_n^2(t-\tau)/a^2}\,d\tau$$

$$-\frac{2A}{a^2}\sum_{n=1}^{\infty}\frac{J_0(\mu_n r/a)}{J_1^2(\mu_n)}\int_0^t\int_0^a w_1(\xi,\tau)\xi J_0(\mu_n\xi/a)\,\mathrm{e}^{-\chi_1\mu_n^2(t-\tau)/a^2}\,d\xi\,d\tau. \tag{6.15}$$

We introduce the notation

$$\alpha = \chi_1\mu_n^2/a^2 - \delta/2. \tag{6.16}$$

According to (6.8) and (6.14),

$$I_1 = \Big|\int_0^t T_1(a,\tau)\,\mathrm{e}^{-\chi_1\mu_n^2(t-\tau)/a^2}\,d\tau\Big| \le \sqrt{\frac{C_5C_7}{2}}\,\mathrm{e}^{-\chi_1\mu_n^2 t/a^2}\begin{cases} \displaystyle\int_0^t \sqrt{\tau}\,\mathrm{e}^{\alpha\tau}\,d\tau, & \delta_1 = \delta, \\ \displaystyle\frac{1}{\sqrt{|\delta_1-\delta|}}\int_0^t |1-\mathrm{e}^{(\delta-\delta_1)\tau}|^{1/2}\,\mathrm{e}^{\alpha\tau}\,d\tau, & \delta_1 \ne \delta. \end{cases}$$

The first integral, obviously, does not exceed the value of the expression $\alpha^{-1}t^{1/2}(\mathrm{e}^{\alpha t}-1)$. For the second integral, we obtain the upper-bound estimate

$$\int_0^t |1-\mathrm{e}^{(\delta-\delta_1)\tau}|^{1/2}\,\mathrm{e}^{\alpha\tau}\,d\tau \le \begin{cases} (\mathrm{e}^{\alpha t}-1)/\alpha, & \delta < \delta_1, \\ (\mathrm{e}^{\alpha_1 t}-1)/\alpha_1, & \delta > \delta_1, \end{cases}$$

where

$$\alpha_1 = \chi_1\mu_n^2/a^2 - \delta_1/2. \tag{6.17}$$

For $\delta > \delta_1$, we use the inequality $(\mathrm{e}^{(\delta-\delta_1)\tau}-1)^{1/2} < \mathrm{e}^{(\delta-\delta_1)\tau/2}$ was used. In view of the notation (6.16) and (6.17) and the obtained estimates, we have the inequality

$$I_1 \le \sqrt{\frac{C_5C_7}{2}}\begin{cases} t^{1/2}(\mathrm{e}^{-\delta t/2} - \mathrm{e}^{-\chi_1\mu_n^2 t/a^2})/\alpha, & \delta_1 = \delta, \\ (\mathrm{e}^{-\delta t/2} - \mathrm{e}^{-\chi_1\mu_n^2 t/a^2})/(\alpha_1\sqrt{\delta_1-\delta}\,), & \delta_1 > \delta, \\ (\mathrm{e}^{-\delta_1 t/2} - \mathrm{e}^{-\chi_1\mu_n^2 t/a^2})/(\alpha_1\sqrt{|\delta_1-\delta|}\,), & \delta_1 < \delta. \end{cases} \tag{6.18}$$

Let us estimate the integral

$$I_2 = \Big|\int_0^t\int_0^a w_1(\xi,\tau)\xi J_0(\mu_n\xi/a)\,\mathrm{e}^{-\chi_1\mu_n^2(t-\tau)/a^2}\,d\xi\,d\tau\Big|.$$

Using (3.25) and the inequality $|J_0(z)| \le 1$, we obtain

$$I_2 \le \frac{a^2 N_1}{2} \begin{cases} (\mathrm{e}^{-\delta t/2} - \mathrm{e}^{-\chi_1 \mu_n^2 t/a^2})/\alpha, & \delta \le 2\nu_1 \mu_1^2/a^2, \\ (\mathrm{e}^{-\nu_1 \mu_1^2/a^2} - \mathrm{e}^{-\chi_1 \mu_n^2 t/a^2})/\alpha_2, & \delta > 2\nu_1 \mu_1^2/a^2, \end{cases} \tag{6.19}$$

where $\alpha_2 = \chi_1 \mu_n^2/a^2 - \nu_1 \mu_1^2/a^2$.

Let us return to the estimate of series (6.15). Since $\mu_n > \mu_1$ for $n > 1$, on the right of inequalities (6.18), (6.19) we can set $n = 1$. Then, a constant $N_2 > 0$ exists such that

$$|T_1(r,t)| \le N_2 t^{1/2} \mathrm{e}^{-\delta_2 t}, \tag{6.20}$$

and the choice of the constant δ_2 depends on the relations among δ, δ_1, $\chi_1 \mu_1^2/a^2$, and $\nu_1 \mu_1^2/a^2$ [see inequalities (6.18) and (6.19)]. In inequality (6.20), the multiplier $t^{1/2}$ appears only if $\delta_1 = \delta$. Thus, we proved the following theorem.

Theorem 2. *If condition* (3.20) *is satisfied, the solution of the initial-boundary-value problem* (6.1) *tends to the zero solution, and estimates* (6.12) *and* (6.20), *are valid which are uniform in the intervals* $[a,b]$ *and* $[0,a]$.

Applying a Laplace transform to problem (6.1) reduces it to the boundary-value problem for the ordinary differential equations

$$\begin{aligned} \tilde{T}_{1rr} + \tilde{T}_{1r}/r - p\tilde{T}_1/\chi_1 &= A\tilde{w}_1/\chi_1, \qquad 0 < r < a, \\ \tilde{T}_{2rr} + \tilde{T}_{2r}/r - p\tilde{T}_2/\chi_2 &= A\tilde{w}_2/\chi_2, \qquad a < r < b; \end{aligned} \tag{6.21}$$

$$\begin{gathered} \tilde{T}_2(b,p) = 0, \qquad |\tilde{T}_1(0,p)| < \infty, \\ \tilde{T}_1(a,p) = \tilde{T}_2(a,p), \qquad k\tilde{T}_{1r}(a,p) = \tilde{T}_{2r}(a,p), \qquad k = k_1/k_2. \end{gathered} \tag{6.22}$$

Here $\tilde{w}_1(r,p)$ and $\tilde{w}_2(r,p)$ are determined from formulas (4.5)–(4.10). The general solutions of Eqs. (6.21) have the form

$$\begin{aligned} \tilde{T}_1(r,p) &= D_1 I_0\Big(\sqrt{\frac{p}{\chi_1}}\, r\Big) - \frac{A\tilde{f}_1(p)}{p^2} - \frac{AC_1}{p\chi_1(1/\chi_1 - 1/\nu_1)}\, I_0\Big(\sqrt{\frac{p}{\nu_1}}\, r\Big), \\ \tilde{T}_2(r,p) &= D_2 I_0\Big(\sqrt{\frac{p}{\chi_2}}\, r\Big) + D_3 K_0\Big(\sqrt{\frac{p}{\chi_2}}\, r\Big) - \frac{A\tilde{f}_2(p)}{p^2} \\ &\quad - \frac{AC_2}{p\chi_2(1/\chi_2 - 1/\nu_2)}\, I_0\Big(\sqrt{\frac{p}{\nu_2}}\, r\Big) - \frac{AC_3}{p\chi_2(1/\chi_2 - 1/\nu_2)}\, K_0\Big(\sqrt{\frac{p}{\nu_2}}\, r\Big) \end{aligned} \tag{6.23}$$

for $\nu_1 \ne \chi_1$, and $\nu_2 \ne \chi_2$, i.e., the Prandtl numbers of the fluids are not equal to unity. From boundary conditions (6.22), we obtain D_1, D_2, and D_3:

$$\begin{gathered} D_1 = \frac{1}{\Delta_1} \begin{vmatrix} H_1 & -I_0(y_1) & -K_0(y_1) \\ H_2 & I_0(z_1) & K_0(z_1) \\ H_3 & -\sqrt{p/\chi_2}\, I_1(y_1) & \sqrt{p/\chi_2}\, K_1(y_1) \end{vmatrix}, \\ D_2 = \frac{1}{\Delta_1} \begin{vmatrix} I_0(x_1) & H_1 & -K_0(y_1) \\ 0 & H_2 & K_0(z_1) \\ k\sqrt{p/\chi_1}\, I_1(x_1) & H_3 & \sqrt{p/\chi_2}\, K_1(y_1) \end{vmatrix}, \\ D_3 = \frac{1}{\Delta_1} \begin{vmatrix} I_0(x_1) & -I_0(y_1) & H_1 \\ 0 & I_0(z_1) & H_2 \\ k\sqrt{p/\chi_1}\, I_1(x_1) & -\sqrt{p/\chi_2}\, I_1(y_1) & H_3 \end{vmatrix}, \\ \Delta_1 = \begin{vmatrix} I_0(x_1) & -I_0(y_1) & -K_0(y_1) \\ 0 & I_0(z_1) & K_0(z_1) \\ k\sqrt{p/\chi_1}\, I_1(x_1) & -\sqrt{p/\chi_2}\, I_1(y_1) & \sqrt{p/\chi_2}\, K_1(y_1) \end{vmatrix}, \end{gathered} \tag{6.24}$$

where $x_1 = a\sqrt{p/\chi_1}$, $y_1 = a\sqrt{p/\chi_2}$, and $z_1 = b\sqrt{p/\chi_2}$,

$$H_1 = \frac{A(\tilde f_1(p) - \tilde f_2(p))}{p^2} + \frac{AC_1}{p\chi_1(1/\chi_1 - 1/\nu_1)}\, I_0\Big(\sqrt{\frac{p}{\nu_1}}\, a\Big)$$

$$- \frac{AC_2}{p\chi_2(1/\chi_2 - 1/\nu_2)}\, I_0\Big(\sqrt{\frac{p}{\nu_2}}\, a\Big) - \frac{AC_3}{p\chi_2(1/\chi_2 - 1/\nu_2)}\, K_0\Big(\sqrt{\frac{p}{\nu_2}}\, a\Big),$$

$$H_2 = \frac{A\tilde f_2(p)}{p^2} + \frac{AC_2}{p\chi_2(1/\chi_2 - 1/\nu_2)}\, I_0\Big(\sqrt{\frac{p}{\nu_2}}\, b\Big) + \frac{AC_3}{p\chi_2(1/\chi_2 - 1/\nu_2)}\, K_0\Big(\sqrt{\frac{p}{\nu_2}}\, b\Big), \tag{6.25}$$

$$H_3 = \frac{kAC_1}{\sqrt{p}\,\chi_1(1/\chi_1 - 1/\nu_1)\sqrt{\nu_1}}\, I_1\Big(\sqrt{\frac{p}{\nu_1}}\, a\Big)$$

$$- \frac{AC_2}{\sqrt{p}\,\chi_2(1/\chi_2 - 1/\nu_2)\sqrt{\nu_2}}\, I_1\Big(\sqrt{\frac{p}{\nu_2}}\, a\Big) + \frac{AC_3}{\sqrt{p}\,\chi_2(1/\chi_2 - 1/\nu_2)\sqrt{\nu_2}}\, K_1\Big(\sqrt{\frac{p}{\nu_2}}\, a\Big).$$

If $\nu_1 = \chi_1$ and $\nu_2 = \chi_2$, solution (6.23) is written as

$$\tilde T_1(r,p) = D_1 I_0\Big(\sqrt{\frac{p}{\nu_1}}\, r\Big) - \frac{A\tilde f_1(p)}{p^2} + \frac{AC_1 r}{2\sqrt{p/\nu_1}\,\chi_1}\, I_1\Big(\sqrt{\frac{p}{\nu_1}}\, r\Big),$$

$$\tilde T_2(r,p) = D_2 I_0\Big(\sqrt{\frac{p}{\nu_2}}\, r\Big) + D_3 K_0\Big(\sqrt{\frac{p}{\nu_2}}\, r\Big) - \frac{A\tilde f_2(p)}{p^2} + \frac{AC_2 r}{2\sqrt{p/\nu_2}\,\chi_2}\, I_1\Big(\sqrt{\frac{p}{\nu_2}}\, r\Big) - \frac{AC_3 r}{2\sqrt{p/\nu_2}\,\chi_2}\, K_1\Big(\sqrt{\frac{p}{\nu_2}}\, r\Big).$$

We show that

$$\lim_{p\to 0} p\tilde T_j(r,p) = T_j^0(r),$$

if $\lim_{p\to 0} p\tilde f_1(p) = f_1^0$, where $T_j^0(r)$ is the steady-state solution (2.1) for temperature perturbations.

We perform the corresponding transformations for $\tilde T_1(r,p)$. Because as $t \to 0$, we have $I_0(t) = 1 + t^2/4 + t^4/64 + O(t^6)$, it follows that

$$p\tilde T_1(r,p) \sim pD_1\Big(1 + \frac{p}{4\chi_1}\, r^2 + \frac{p^2}{64\chi_1^2}\, r^4\Big) - \frac{A\tilde f_1}{p} - \frac{AC_1}{1 - \chi_1/\nu_1}\, \Big(1 + \frac{p}{4\nu_1}\, r^2 + \frac{p^2}{64\nu_1^2}\, r^4\Big)$$

$$= pD_1 - \frac{A\tilde f_1}{p} - \frac{AC_1}{1 - \chi_1/\nu_1} + \Big(\frac{p^2 D_1}{4\chi_1} - \frac{AC_1 p}{4\nu_1(1 - \chi_1/\nu_1)}\Big) r^2 + \Big(\frac{p^3 D_1}{64\chi_1^2} - \frac{AC_1 p^2}{64\nu_1^2(1 - \chi_1/\nu_1)}\Big) r^4 \tag{6.26}$$

as $p \to 0$. Hence, it is necessary to investigate the behavior of the functions H_1, H_2, and H_3 in (6.25) and D_1 in (6.24) as $p \to 0$.

Using the asymptotic representations (4.12) for $p \to 0$, we obtain

$$H_1 \sim \frac{A\tilde f_1}{p^2}\Big(\frac{\rho\chi_2}{\nu_2 - \chi_2} - \frac{\chi_1}{\nu_1 - \chi_1} + \frac{(\chi_2\nu_1 - \chi_1\nu_2)\rho(y^2 - z^2)}{4(\nu_1 - \chi_1)(\nu_2 - \chi_2)}$$

$$- \frac{\rho\nu_2 y^2}{8(\nu_2 - \chi_2)}\, [x^2 + (\rho - 1)z^2 - \rho y^2] \ln y\Big),$$

$$H_2 \sim -\frac{A\rho\tilde f_1\chi_2}{p^2(\nu_2 - \chi_2)}\Big[1 - \frac{\nu_2}{8}\Big(\frac{z^4}{2} + y^2(x^2 + (\rho - 1)z^2 - \rho y^2) \ln z\Big)\Big], \tag{6.27}$$

$$H_3 \sim \frac{A\tilde f_1\sqrt{\nu_1}}{2p\sqrt{p}}\, \Big[- \frac{k}{\nu_1 - \chi_1} + \frac{\rho}{\nu_2 - \chi_2} + \frac{k}{4(\nu_1 - \chi_1)}\Big(\frac{x^2}{2} + \rho(z^2 - y^2)\Big)$$

$$+ \frac{\rho}{4(\nu_2 - \chi_2)}\Big(x^2 + \rho z^2 - \frac{(2\rho + 1)y^2}{2}\Big)\Big] x.$$

From (6.24), we have

$$\Delta_1(p) \sim \frac{1}{a}\Big[1 + \frac{x_1^2 + z_1^2}{4} + \frac{y_1^2}{2}\Big(1 - \frac{k}{\chi}\Big)\ln\frac{a}{b}\Big]. \tag{6.28}$$

To calculate the coefficient for r^4 in (6.26), it is sufficient to use only the first terms of the expansion in all formulas (6.27). Indeed, using (4.12), (6.27) and (6.28), (6.24), we have

$$D_1 \sim \frac{A\tilde{f}_1}{p^2}\Big(\frac{\rho\chi_2}{\nu_2-\chi_2} - \frac{\chi_1}{\nu_1-\chi_1}\Big) - \frac{A\tilde{f}_1}{p^2}\,\frac{\rho\chi_2}{\nu_2-\chi_2} = -\frac{A\tilde{f}_1\chi_1}{p^2(\nu_1-\chi_1)}; \tag{6.29}$$

therefore,

$$\frac{p^3D_1}{64\chi_1^2} - \frac{AC_1p^2}{64\nu_1^2(1-\chi_1/\nu_1)} \sim \frac{pA\tilde{f}_1}{64}\Big(-\frac{1}{\chi_1(\nu_1-\chi_1)} + \frac{1}{\nu_1(\nu_1-\chi_1)}\Big) = -\frac{pA\tilde{f}_1}{64\chi_1\nu_1}.$$

For $p \to 0$, the last expression is the coefficient at r^4 in formula (2.1) for $T_1^0(r)$.

For $p \to 0$, let us find the limit of the coefficient at r^2 in formula (6.26). We note that in (6.26), the terms of the zero order in p are mutually cancelled. Then, from (6.26), (4.12), and (6.29), we obtain

$$\frac{p^2D_1}{4\chi_1} - \frac{pAC_1}{4(\nu_1-\chi_1)} \sim -\frac{p^2A\tilde{f}_1\chi_1}{4\chi_1p^2(\nu_1-\chi_1)} + \frac{pA\tilde{f}_1}{4p(\nu_1-\chi_1)} \equiv 0.$$

Consequently, in all expansions, it is necessary to keep terms of the order x_j^2, y_j^2, and z_j^2 ($j = 1, 2$) that correspond to the order p. As a result, we obtain the required expression $p\tilde{f}_1(p)$. Using relations (4.12) for C_1, C_2, and C_3, relations (6.27) for H_1, H_2, and H_3, expression (6.29) for D_1, and relation (6.28), we obtain

$$\frac{p^2D_1}{4\chi_1} - \frac{pAC_1}{4(\nu_1-\chi_1)} \sim \frac{pA\tilde{f}_1(p)}{16\chi_1}\Big[\frac{\rho\chi_2}{\nu_2-\chi_2}\Big(\frac{a^2}{\nu_2} - \frac{a^2}{\chi_2} + \frac{b^2}{\chi_2} - \frac{b^2}{\nu_2}\Big) + \frac{\chi_1}{\nu_1-\chi_1}\Big(\frac{a^2}{\chi_1} - \frac{a^2}{\nu_1}\Big)\Big] = \frac{Ap\tilde{f}_1(p)}{16\nu_1\chi_1}\,[a^2 + \mu(b^2-a^2)]$$

as $p \to 0$. As $p \to 0$, this expression is the coefficient at r^2 in (6.26).

Using the asymptotic relations (6.27) for $p \to 0$, the free term in (6.26) is expressed as

$$pD_1 - \frac{A\tilde{f}_1}{p} - \frac{A\nu_1C_1}{\nu_1-\chi_1} \sim \frac{a^4f_1^0A}{8\nu_1\chi_1}\Big\{-\frac{3\mu\chi}{8}\Big(\frac{b}{a}\Big)^4$$

$$+ \Big[k\Big(\frac{1}{2} + \mu\Big(\frac{b^2}{a^2} - 1\Big)\Big) + \mu\chi\Big(\frac{1}{2} - \frac{b^2}{a^2}\Big)\Big]\ln\frac{a}{b} - \frac{\mu}{2}\Big[\frac{b^2}{a^2} - 1 + \chi\Big(\frac{1}{4} - \frac{b^2}{a^2}\Big)\Big] - \frac{3}{8}\Big\},$$

which coincides with the expression for the constant C_1 in formula (2.1) for $T_1^0(r)$.

To study the behavior of the velocities $w_j(r,t)$ and temperature perturbations $T_j(r,t)$ for an arbitrary specified pressure gradient $f_1(t)$, it is necessary to employ the numerical method of inversion of the Laplace transform.

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